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KINETIC APPROACH TO SOLITON TURBULENCE

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ABSTRACT

The forced Schrödinger equation for strong fluctuations describes the micro-hydrodymical state of soliton turbulence. It is valid for large-scale turbulence where the internal waves can interact with the velocity fluctuations. It was derived by Tchen, as a fluid analogue, from the Navier-stokes equations for compressible flows. It is transformed into a master equation, to be subsequently decomposed into a macro-group, a micro-group, and a submicro-group, representative of the three transport processes of spectral evolution, transport property, and relaxation. The loss of memory in formulating the relaxation function gives the closure so that the transport property can attain its equilibrium. The kinetic equation for the macro-distribution is returned to the continuum by the method of moments to derive the equation of spectral evolution. The spectral flow is governed by the three transport functions as follows. 1) A transfer function from the (cubic) modulational nonlinearity forms a direct cascade. 2) A transfer function from the driving force, which enters as a convective nonlinearity in the phase-space, produces

a reverse cascade for accumulating the solitons toward the large-scale end of the spectrum. 3) A coupling function represents the excitation of the solitons by the driving force which acts as a scattering (or emission) from velocity fluctuations.

The following power laws

$$E_k^2 \sim k^{-m} \quad , \qquad N_k^2 \sim k^{-n}$$

are found for the soliton field intensity and density, respectively, with

$$m = 0 \ , \qquad n = 0$$

in the inertia subrange by reverse cascade, and

$$m = 2 \ . \qquad n = 4$$

in the coupling subrange at larger wavenumbers. These analytical predictions are verified by experiments. The forced Schrödinger equation is valid for fluids and plasmas.

1. Introduction: The Forced Schrödinger Equation for the Description of Soliton Turbulence

The multi-scale motions in fluids and plasmas are more adequately described by the nonlinear Schrödinger equation[1]

$$(\partial_t - i\nu_n \nabla^2 + i\frac{1}{2}\omega_n \tilde{A}\tilde{N})\tilde{\underline{E}} = 0 \quad , \tag{1}$$

instead of the usual Navier-Stokes equation. The modulation relation

$$\tilde{N} = -\tilde{A}\lambda|\tilde{\underline{E}}|^2 \tag{2}$$

makes the Schrödinger equation cubically nonlinear. Here $\tilde{\underline{E}}$ is the longitudinal field and is related to the density fluctuation $\tilde{N}$ by (2). The density is normalized such that

$$\bar{N} = 1 \ . \tag{3}$$

The over-bar denotes an ensemble average, $\tilde{()}$ denotes a fluctuation, and $\tilde{A}$ is the operator. the constants are

$$\nu_n \ , \quad \omega_n \ , \quad \lambda \ ,$$

having the dimensions of viscosity, frequency, and $(\text{acceleration})^{-2}$, respectively.

The representation of the many-scale motions by the Schrödinger equation is well recognized in plasmas, because of the presence of the ion and electron species, and therefore the Schrödinger equation was first derived for plasmas. The analogous representation for fluids, called the fluid analogue, is not evident. In fluids of large-scale motions where the compressibility is not negligible, the longitudinal density waves may couple with the transverse velocity fluctuations. The internal gravity waves of large scales may also couple with the velocity fluctuations and transmit the effects to smaller-scale turbulence. These fluid motions of many scales justify the fluid analogue of the Schrödinger equation and its derivation from the Navier-Stokes equation for compressible fluids. The reason for having a fluid analogue lies in the fact that the Navier-Stokes equation is formally tied down to small scales by the nonlinearity $\hat{\underset{\sim}{u}} \cdot \nabla \hat{\underset{\sim}{u}}$ of the gradient type, where $\hat{\underset{\sim}{u}}$ is the fluid velocity. The Schrödinger equation has the cubic nonlinearity $|\tilde{\underset{\sim}{E}}|^2 \tilde{\underset{\sim}{E}}$ not of the gradient type, and is found more suitable for the description of large scales.

An examination of (1) reveals that the modulational nonlinearity yields a direct cascade, and should require a sink for balancing the spectral flow. An artificial sink would introduce an ambiguity. Hence the Schrödinger equation in the original form (1) cannot describe turbulence. The reason for this difficulty lies in that the original derivation was intended for weak fluctuations. The fluid analogue for strong turbulence requires a new derivation.

We have derived a generalized Schrödinger equation in the form as follows:[2]

$$(\partial_t - i\gamma_n \nabla^2 + i\tfrac{1}{2}\omega_n \tilde{A}\,\tilde{N})\tilde{E} = -i\tilde{X} , \tag{4}$$

with a driving force $\tilde{X}$, that obeys the Poisson equation

$$\nabla \tilde{X} = -\tfrac{1}{2}\omega_n \tilde{r} , \tag{5}$$

and contains the scattering (or emission)[3]

$$\tilde{r} = \tilde{A}\,\nabla\nabla : \tilde{u}\tilde{u} . \tag{6}$$

by the velocity fluctuation $\tilde{u}$. The equation (4) will be called the "forced Schrödinger equation". We consider isotropy by assuming

$$\bar{E} = 0, \quad \bar{X} = 0 . \tag{7}$$

In conclusion, the forced schrödinger equation, but not the standard Schrödinger equation and the Zakharov equations, will develop turbulence.

2. Master Equation for Solitons

For the evolution of the spectral distribution of field fluctuations, the knowledge of the eddy transport properties is necessary, for which a kinetic theory is most appropriate, like in any transport theories of gases and plasmas. A continuum theory of correlations, as based upon the forced Schrödinger equation, would immediately be faced with the difficulty of high-order correlations. The most well known kinetic method for treating collective phenomena is the statistial method of many bodies by Bogoliubov. It introduces a master equation of N particles that interact with a field, and by successive intergrations, generates a hierarchy of equations for the many-particle distribution functions, called the BBGKY hierarchy. For the spectral evolution, the closure that leads to the kinetic equation for the pair-distribution is necessary. This is considered to be a difficult task in statistical mechanics. In addition, the field of interaction $\hat{\underline{E}}(t,\underline{x})$ that is a simple Coulomb field in plasmas becomes much more complicate in solitons, as having to satisfy the nonlinear partial differential equation (4) in view of its self-consistency. We also wish to avoid the difficulty from the kinetic equation of pair-distribution function.

For these reasons, we devise a group-kinetic method. The forced Schrödinger equation (4) can be considered as the first moment of the following master equation

$$(\partial_t + \tilde{A}\,\tilde{L})\tilde{f}(t,\underline{x},\underline{E}) = 0 \ . \tag{8}$$

The differential operator

$$\hat{L} = L_\nu + \tilde{L}_N + \tilde{L}_X \tag{9}$$

consists of three components

$$L_\nu = -i\nu_n \nabla^2, \ \tilde{L}_N = i\tfrac{1}{2}\omega_n \tilde{N}, \quad L_X = -i\tilde{X}\cdot\partial, \quad \partial = \partial/\partial E \ . \tag{10}$$

The function

$$\hat{f}(t,x,E) = \delta\left[E - \hat{E}(t,x)\right] \tag{11}$$

in the phase space has E as an independent variable. The zeroth moment gives

$$\int dE\, \hat{f} = 1, \int dE\, \bar{f} = 1, \int dE\, \tilde{f} = 0, \tag{12a}$$

and the first moment gives

$$\int dE\, E\, \hat{f} = \hat{E}, \quad \bar{E} = \int dE\, E\, \bar{f}\ , \quad \tilde{E} = \int dE\, E\, \tilde{f} \ . \tag{12b}$$

The master equation in the form (8) describes the microdynamical state of turbulence. This can be shown by reverting it into the forced Schrödinger equation through the moment operation.

Since the field amplitude $\tilde{E}$ is complex, the distribution function $\tilde{f}(t,x,E)$ will be complex too, while the functions $\tilde{N}$ and $\tilde{X}$ are real.

3. Kinetic Equation of Soliton Turbulence

For soliton turbulence, we need to derive a kinetic equation that explicitly shows the interaction between the large and small scales. Since both the forced Schrödinger equation and the master equation (8) as a kinetic correspondent describe the microdynamical state of soliton turbulence in too many details that are unnecessary for a statistical treatment, a procedure of coarse-graining is needed. The decomposition into Fourier components will not suffice. In compliance with the separate statistical roles, we decompose the fluctuation

$$\tilde{()} = ()^{\circ} + ()' \tag{13a}$$

into a macro-group $()^{\circ}$ and a micro-group $()'$, and by re-scaling the micro-group

$$()' = ()^{1} + ()'' \tag{13b}$$

into a submacro-group $()^{1}$ and a submicro-group $()''$. The three groups with operators

$$A^{\circ}, \quad A^{1}, \quad A'' \tag{14}$$

represent the three transport processes of spectral evolution, transport property, and relaxation. The loss of memory in the relaxation makes the transport coefficients approach their equilibrium, and yields the closure. The three groups are separated by the three durations of correlations

$$\tau_c^{[o]} > \tau_c^{[\prime]} > \tau_c^{[\prime\prime]} . \tag{15}$$

The superscripts

$$()^{[o]} , \quad ()^{[\prime]} , \quad ()^{[\prime\prime]} \tag{16}$$

with square brackets denote the deterministic statistical properties as derived from the fluctuations (13) and from ensemble averages.

We scale the master equation by the operators A° and A^{1} to obtain the system of equations

$$(\partial_t + A^{\circ} \hat{L} - \mathcal{C}^{[1]}) f^{\circ} = - L^{\circ} \bar{f} \tag{17}$$

and

$$(\partial_t + A' \hat{L} - \mathcal{C}^{[\prime\prime]}) f^{1} = - L'(\bar{f} + f^{\circ}) . \tag{18}$$

The eddy collisions are:

$$\mathcal{C}^{[1]} f^{\circ} = - A^{\circ} L^{1} f^{1} \tag{19a}$$

$$\mathcal{C}^{[\prime\prime]} f^{1} = - A^{1} L'' f'' . \tag{19b}$$

The submicro-distribution

$$f'' = f^{(2)} + f^{(3)} + \ \ldots \qquad (20)$$

is a cluster of high-order groups that represents the frictional environment in which f^1 evolves. The collision coefficients $\mathcal{C}^{[1]}$ and $\mathcal{C}^{['']}$ may act as integral operators.

The composition of $\mathcal{C}^{[1]}$ is determined by (18). Upon integrating, we have the distribution function

$$f^1 = - \int_0^t d\tau\, \Lambda^1(t,t-\tau) L^1(t-\tau)\ (\bar{f}+f^{\circ})_{t-\tau} \ , \qquad (21a)$$

from which we calculate the eddy collision

$$- A^{\circ} L^1 f^1 = A^{\circ} \int_0^t d\tau\ L^1(t) \Lambda^1(t,t-\tau)\ L^1(t-\tau)(\bar{f}+f^{\circ})_{t-\tau} . \qquad (21b)$$

Since the groups

$$A^1 \quad \text{and} \quad A^{\circ}$$

have the time scales

$$\tau_c^{[1]} \quad \text{and} \quad \tau_c^{[\circ]} \ ,$$

by (15), we can replace the upper limit t of integration by ∞ without loss of generality, and obtain an asymptotic collision coefficient

$$\mathcal{C}^{[1]} = \int_0^{t\to\infty} d\tau \left\langle L^1(t)\ \Lambda^1(t,t-\tau)\ L^1(t-\tau)\right\rangle \tag{22a}$$

that is deterministic. Here we have written

$$L^1(t) = L^1(t,\underset{\sim}{x},\underset{\sim}{E}) \quad .$$

for the sake of simplicity, and Λ^1 is the operator of evolution. In an analogous manner, we define

$$\mathcal{C}^{['1]} = \int_0^{t\to\infty} d\tau \left\langle L'(t)\ \Lambda^1(t,t-\tau)\ L'(t-\tau)\right\rangle . \tag{22b}$$

By noting

$$- A^{\circ}\, L'L' = - A^{\circ}(L^1L^1 + L^{(2)}L^{(2)} + \ \ldots\ ) \ ,$$

we can show the equivalence between (22a) and (22b), and write

$$\mathcal{C}^{[1]} = \mathcal{C}^{['1]} . \tag{23}$$

With the collision coefficient thus determined in the form (22b), the equation of evolution of the macro-distribution (17) becomes

$$(\partial_t + A^\circ \hat{L}) f^\circ = - L^\circ \bar{f} + \mathcal{C}^{[1]} f^\circ \quad . \tag{24}$$

It is explicit in f°, and is called the kinetic equation of turbulence.

4. Schrödinger Equation of Soliton Turbulence

By taking the moment of the kinetic equation (24), we revert to the continuum and derive the following Schrödinger equation of soliton turbulence:

$$(\partial_t - i \nu_n \nabla^2 + i \tfrac{1}{2} \omega_n A^\circ N^\circ) \underset{\sim}{E}^\circ = - i \underset{\sim}{X}^\circ + \underset{\sim}{J}^\circ , \tag{25}$$

with

$$\underset{\sim}{J}^\circ = \int d\underset{\sim}{E} \, \underset{\sim}{E} \, \mathcal{C}^{[1]} f^\circ \quad , \tag{26a}$$

or equivalently

$$J^\circ = \int dE \, E \, \mathcal{C}^{[1]} f^\circ \quad , \tag{26b}$$

by (23). It has all the constituents of the original forced Schrödinger equation (4), and, in addition, the eddy stress (26). The latter represents the statistical effects of the colliding micro-eddies.

By writing

$$\mathcal{C}^{[1]} = \mathcal{C}_N^{[1]} + \mathcal{C}_X^{[1]}, \tag{27}$$

into two components as corresponding to L_N' and L_X', we transform the eddy stress (26b) into two parts as follows:

$$J^{\circ} = J_N^{\circ} + J_X^{\circ}, \tag{28a}$$

with

$$J_N^{\circ} = \int d\underset{\sim}{E}\, \underset{\sim}{E}\, \mathcal{C}_N^{[1]} f^{\circ}, \qquad J_X^{\circ} = \int d\underset{\sim}{E}\, \underset{\sim}{E}\, \mathcal{C}_X^{[1]} f^{\circ}. \tag{28b}$$

The limits of integration are understood to extend from $-\infty$ to ∞. The coefficients of collision are integral operators, and the operator symbol $\{\,\}$ is understood.

5. Spectral Flow

Upon multiplying (25) by $\underset{\sim}{E}^{\circ *}$ and adding the complex conjugate part, we obtain the energy equation for the solitons, as follows:

$$\frac{1}{2}\partial_t\left\langle|\underset{\sim}{E}^{\circ}|^2\right\rangle = W^{[\circ]} - T^{[\circ]}$$

$$= -T_X^{[\circ]} + W^{[\circ]} - T_N^{[\circ]}. \tag{29}$$

Homogeneity and isotropy are assumed.

The transport functions are written in the order of their importance with increasing wavenumbers.

The coupling function

$$W^{\circ} = \frac{1}{2}\left[-i\left\langle\underset{\sim}{X}^{\circ}\cdot\underset{\sim}{E}^{\circ *}\right\rangle + i\left\langle\underset{\sim}{X}^{\circ}\,\underset{\sim}{E}^{\circ}\right\rangle\right] \tag{30}$$

excites the solitons by the driving force. The transfer function

$$T_X^{[\circ]} = -\left\langle\left|\underset{\sim}{J}_X^{\circ}\cdot\underset{\sim}{E}^{\circ *}\right|\right\rangle \tag{31a}$$

for the reverse cascade accumulates the solitons toward the large-scale end of the spectrum, and the transfer function

$$T_N^{[\circ]} = -\left\langle\left|\underset{\sim}{J}_N^{\circ}\cdot\underset{\sim}{E}^{\circ *}\right|\right\rangle \tag{31b}$$

for the direct cascade causes a disintegration of the large eddies into smaller eddies. The vertical bars represent the absolute values. The remaining convective terms

$$(- i \nu_n \nabla^2 + i \tfrac{1}{2}\omega_n A^\circ N^\circ) \underset{\sim}{E}^\circ \tag{32}$$

in the left hand side of (25) do not contribute to the spectral flow, on account of their imaginary character.

6. Theory of Eddy Transport

6.1. Eddy diffusivities

It will be convenient to introduce the following eddy diffusivity operators

$$K_N^{[1]} = \int_0^{t\to\infty} d\tau \left\langle N^1(t)\, \Lambda^1(t,t-\tau)\, N^1(t-\tau) \right\rangle \tag{33a}$$

$$\underset{\approx}{K}_X^{[1]} = \int_0^{t\to\infty} d\tau \left\langle \underset{\sim}{X}^1(t)\, \Lambda^1(t,t-\tau)\, \underset{\sim}{X}^1(t-\tau) \right\rangle , \tag{33b}$$

with the evolution operator

$$\Lambda^1 = A^1 \Lambda , \tag{34}$$

for transforming the collision coefficients into

$$\mathcal{C}_N^{[1]} = -\left(\tfrac{1}{2}\omega_n\right)^2 K_N^{[1]} \tag{35a}$$

$$\mathcal{C}_X^{[1]} = - \underset{\sim}{\partial} \cdot \underset{\approx}{K}_X^{[1]} \cdot \underset{\sim}{\partial} , \tag{35b}$$

and the transfer functions into

$$T_N^{[o]} = -\int d\underset{\sim}{E}\ \underset{\sim}{E}\cdot \mathcal{T}_N^{[1]} \left\langle \left| \underset{\sim}{E}^{o*} f^o \right| \right\rangle$$

$$= \left(\tfrac{1}{2}\omega_n\right)^2 \int d\underset{\sim}{E}\ \underset{\sim}{E}\cdot K_N^{[1]} \left\langle \left| \underset{\sim}{E}^{o*} f^o \right| \right\rangle \qquad (36a)$$

$$T_X^{[o]} = -\int d\underset{\sim}{E}\ \underset{\sim}{E}\cdot \mathcal{T}_X^{[1]} \left\langle \left| \underset{\sim}{E}^{o*} f^o \right| \right\rangle$$

$$= \int d\underset{\sim}{E}\ \underset{\sim}{E}\ \underset{\sim}{\partial}\ K_X^{[1]}\cdot \underset{\sim}{\partial} \left\langle \left| \underset{\sim}{E}^{o*} f^o \right| \right\rangle . \qquad (36b)$$

Use is made of (28), (31) and (35). The approximation retains those components of diffusivity (33) originating from the auto-correlations, and neglects others from cross-correlations.

The diffusivities upon which the transfer functions depend are the τ-integrations of the Lagrangian correlations, the orbit being characterized by the fluctuating evolution operator Λ^1. For the sake of simplicity and as part of our attempt of memory loss, we approximate the operator by its average

$$\Lambda^1 \simeq \Lambda^{[1]} ,$$

and write the diffusivities (33) in the Fourier form, as follows

$$K_N^{[1]} = \int d\underset{\sim}{k}'' \chi \langle N'(\underset{\sim}{k}'')\ N'(-\underset{\sim}{k}'') \rangle \tau_k^{['']}$$

$$= N_k^{[1]2} \left\{ \tau_k^{['']} \right\} \qquad (37a)$$

$$K_X^{[1]} = \frac{1}{3} \int d\underset{\sim}{k}'' \chi \langle \underset{\sim}{X}'(\underset{\sim}{k}'') \cdot \underset{\sim}{X}'(-\underset{\sim}{k}'') \rangle \tau_k^{['']}$$

$$= \frac{1}{3} X_k^{[1]2} \left\{ \tau_k^{['']} \right\} . \qquad (37b)$$

Here

$$\chi \equiv (\pi / M)^{''}$$

is the coefficient of Fourier truncation in three dimensions within a length interval 2M which may be as large as desired, and

$$N_k^{[1]\,2} = \int d\underset{\sim}{k}'' \chi \left\langle N'(\underset{\sim}{k}'')\ N'(-\underset{\sim}{k}'') \right\rangle \qquad (38a)$$

$$X_k^{[1]\,2} = \int d\underset{\sim}{k}'' \chi \left\langle \underset{\sim}{X}'(\underset{\sim}{k}'') \cdot \underset{\sim}{X}'(-\underset{\sim}{k}'') \right\rangle \qquad (38b)$$

are the spectral intensities of $\tilde{N}$- and $\tilde{\underset{\sim}{X}}$- fluctuations, such that

$$N_{k=0}^{[1]\,2} = \langle \tilde{N}^2 \rangle , \qquad X_{k=0}^{[1]\,2} = \langle \tilde{\underset{\sim}{X}}^2 \rangle .$$

The intensities (38) serve as operators.

The submicro-time

$$\tau_k^{[\text{"}]} = \int_0^{t\to\infty} d\tau \; h^{[\text{"}]}(\tau) \tag{39}$$

is the relaxation time for the approach of the diffusivities (37) to equilibrium. The same relaxation holds for both diffusivities, since they share the identical operator of evolution Λ^1 . The orbit function is

$$\begin{aligned} h^{[\text{"}]}(\tau) &= h_E(\tau)\; h_N^{[\text{"}]}(\tau)\; h_X^{[\text{"}]}(\tau) \\ &\cong h_E(\tau)\; h_N^{[\text{"}]}(\tau) \end{aligned} \tag{40}$$

with the components:

$$h_E(\tau) = \exp(-\tfrac{1}{2}\, i\underset{\sim}{k}'' \cdot \underset{\sim}{E}\; \tau^2) \tag{41a}$$

$$h_N^{[\text{"}]}(\tau) = \exp\; \mathcal{C}_N^{[\text{"}]} \;. \tag{41b}$$

The first component relates to a streaming by $\underset{\sim}{E}$ in weak turbulence, and the second component relates to an internal modulation by $\mathcal{C}_N^{[\text{"}]} < 0$ in strong turbulence. The third component

$$h_X^{[\text{"}]}(\tau) = \exp\, \mathcal{C}_X^{[\text{"}]}\, \tau$$

is omitted in view of its externol and divergent character.

The orbit function is the result of integrqtion of the equation (18) for f^1 by the use of the evolution operqtor Λ^1 thct entails $\zeta^{["]}$

From the characteric equations of the partial differential master equation, one can determine the exact orbit with more orbital components than those listed in (41a) and (41b). We have selected the major ones that represent streaming and modulation only. With the assumption of similarity between

$$\zeta_N^{["]} \quad \text{and} \quad \zeta_N^{[']} , \tag{42}$$

a loop is formed, causing a loss of memory and yielding a closure.

Upon substituting (40) into (39), and subsequently into (37), (35) and (36), we transform the diffusivities into

$$K_N^{[']} = N_k^{[']2} \left\{ \int_0^{t\to\infty} d\tau \; h_E(\tau)\, h_N^{["]}(\tau) \right\} \tag{43a}$$

$$K_X^{[']} = \frac{1}{3} X_k^{[']2} \left\{ \int_0^{t\to\infty} d\tau \; h_E(\tau)\, h_N^{["]}(\tau) \right\} , \tag{43b}$$

and the transfer functions into

$$T_N^{[\circ]} \cong \left(\frac{1}{2}\omega_n\right)^2 N_k^{[']2} \left\{ \int_0^{t\to\infty} d\tau \; \Theta_k^{[\circ]}(\tau) \right\} \tag{44a}$$

$$T_X^{[\circ]} \cong - \frac{1}{3} \; X_k^{[']2} \left\{ \frac{k''^2}{4} \int_0^{t\to\infty} d\tau \; \tau^4 \, \Theta_k^{[\circ]}(\tau) \right\} , \tag{44b}$$

where

$$\Theta_k^{[\circ]}(\tau) \equiv \int d\underset{\sim}{E}\ \underset{\sim}{E} \cdot h_E(\tau)\ h_N^{[\shortparallel]}(\tau) \left\langle \left| \underset{\sim}{E}^{\circ *} f^{\circ} \right| \right\rangle \qquad (44c)$$

is the modified energy, as obtained by weighting the field energy

$$\int d\underset{\sim}{E}\ \underset{\sim}{E} \cdot \left\langle \left| \underset{\sim}{E}^{\circ *} f^{\circ} \right| \right\rangle = \left\langle \underset{\sim}{E}^{\circ *} \underset{\sim}{E}^{\circ} \right\rangle$$

by the orbit function.

By noting that

$$\partial^2 h_E(\tau) = - \frac{k_{\shortparallel}^{2}}{4} \tau^4\ h_E(\tau) \quad , \qquad (45a)$$

and by partial integration, we can write

$$\underset{\sim}{\partial} \cdot h_E(\tau)\ h_N^{[\shortparallel]}(\tau)\ \underset{\sim}{\partial} = - \frac{k_{\shortparallel}^2}{4} \tau^4 \quad h_E(\tau)\ h_N^{[\shortparallel]}(\tau) \qquad (45b)$$

and

$$\int_0^{t\to\infty} d\tau\ \ \underset{\sim}{\partial} \cdot h_E(\tau)\ h_N^{[\shortparallel]} \underset{\sim}{\partial} = - \frac{k_{\shortparallel}^2}{4} \int_0^{t\to\infty} d\tau\, \tau^4\ h_E(\tau) h_N^{[\shortparallel]}(\tau) .$$

(45c)

The approximation of keeping the highest moment in τ has been used in (44a) and (44b).

The change of signs in the two transfer functions (44a) and (44b) indicates that $T_N^{[o]}$ is a direct cascade and that $T_X^{[o]}$ is a reverse cascade.

6.2. Fluidization as a means of closure

We recall that the decomposition into the three groups with operators

$$\overset{\circ}{A}\ ,\ A'\ ,\ A''$$

corresponds to a decomposition into three transport processes of evolution, diffusivity, and relaxation. Firstly, the macro-scaling yields the spectral evolution (29) with its coupling function $W^{[o]}$ and the transfer functions $T_X^{[o]}$, $T_N^{[o]}$. By definition (30), the coupling function governs the spectral flow between the two macro-groups $\overset{\circ}{\underset{\sim}{X}}$ and $\overset{\circ}{\underset{\sim}{E}}$. By definitions (31a), the transfer function $T_X^{[o]}$ governs the spectral flow from $\underset{\sim}{X}'$ into $\overset{\circ}{\underset{\sim}{E}}$ through the intemediary of the collision coefficient $\mathcal{C}_X^{[']}$. Similarly by definition (31b), the transfer function $T_N^{[o]}$ governs the spectral flow from $\overset{\circ}{\underset{\sim}{E}}$ into N' through the intermediary of the collision coefficient $\mathcal{C}_N^{[']}$. Secondly, the collision coefficients are proportional to the diffusivities $K_X^{[']}$ and $K_N^{[']}$ by (35). Thirdly, the approach to equilibrium and hence the closure depend on the time of relaxation $\tau_k^{['']}$, which enters into the diffusivities by (37). We have determined the relaxation by analyzing the orbit function in two components $h_E(\tau)\ h_N^{['']}(\tau)$ from (40) and (41). They control the equilibrium of the diffusivities (43) and the transfer functions (44).

In strong turbulence the streaming by $\underset{\sim}{E}$ is negligible, giving

$$h_E(\tau) = 1 . \tag{46a}$$

The cluster of distribution functions (20) which shapes the frictional property of the medium in which f^1 evolves is now assumed to act like a fluid with a collision coefficient

$$\mathcal{C}_{Nf}^{[\prime\prime]}(t,\underset{\sim}{x}) = \mathcal{C}_N^{[\prime\prime]}(t,\underset{\sim}{x},\underset{\sim}{E})\Big|_{\underset{\sim}{E}=0} . \tag{46b}$$

This means that the collision coefficient $\mathcal{C}_N^{[\prime\prime]}(t,\underset{\sim}{x},\underset{\sim}{E})$ in the kinetic representation with its individuality in $\underset{\sim}{E}$ as an independent variable will be reduced into a collision coefficient $\mathcal{C}_{Nf}^{[\prime\prime]}(t,\underset{\sim}{x})$ in the fluid representation by losing the individuality in $\underset{\sim}{E}$. In this way, $\mathcal{C}_{Nf}^{[\prime\prime]}$ ceases to be an operator. Hence the "fluidization" of the cluster of distributions yields the closure.

6.3. <u>Calculation of the collision coefficients, the diffusivities, and the transfer functions</u>

With the approximation of strong turbulence (46a) and the hypothesis of fluidization of the cluster (46b), we reduce (41b) and (44c) into

$$h_N^{[\prime\prime]} = \exp \mathcal{C}_{Nf}^{[\prime\prime]} \tau \tag{47}$$

and

$$\Theta_k^{[0]}(\tau) = h_E(\tau)\, h_N^{[''] }(\tau) \left\langle |\underset{\sim}{E}^{\circ}|^2 \right\rangle . \tag{48}$$

At the same time we calculate the moments in τ

$$\int_0^{\infty} d\tau \; h_E(\tau)\, h_N^{['']}(\tau) = \left| \zeta_{Nf}^{['']} \right|^{-1} \tag{49a}$$

$$\int_0^{\infty} d\tau \; \tau^4 \, h_E(\tau)\, h_N^{['']}(\tau) = c_X \left| \zeta_{Nf}^{['']} \right|^{-5} , \tag{49b}$$

with $c_X = 4!$, to transform (42), (35) and (44) into

$$K_N^{[']} = N_k^{[']\,2} \left\{ \left| \zeta_{Nf}^{['']} \right|^{-1} \right\} \tag{50a}$$

$$K_X^{[']} = \frac{1}{3} X_k^{[']\,2} \left\{ \left| \zeta_{Nf}^{['']} \right|^{-1} \right\} , \tag{50b}$$

$$\zeta_{Nf}^{[']} = -\left(\frac{1}{2}\omega_n\right)^2 N_k^{[']2} \left| \zeta_{Nf}^{['']} \right|^{-1} \tag{51a}$$

$$\zeta_{Xf}^{[']} = \frac{1}{3} X_k^{[']\,2} \left\{ \frac{k''^2}{4} c_X \left| \zeta_{Nf}^{['']} \right|^{-5} \right\} . \tag{51b}$$

and

$$T_{Nf}^{[0]} = -\, \zeta_{Nf}^{[']} \; E_k^{[0]2} \tag{52a}$$

$$T_{Xf}^{[0]} = -\, \zeta_{Xf}^{[']} \; E_k^{[0]\,2} , \tag{52b}$$

respectively. The subscript $(\;)_f$ denotes a fluidized property.

By writing the spectral intensity in terms of the integral of the spectral density, we have

$$N_k^{[']2} = 2\int_k^\infty dk''\ F_N(k'')$$

$$X_k^{[']2} = 2\int_k^\infty dk''\ F_X(k'')\ , \tag{53}$$

where $F_N(k'')$ and $F_X(k'')$ are spectral densities of the $\tilde{N}$- and $\tilde{X}$- fluctuations.

By the use of the notation (53), we can rewrite (5la) in the form of the following integral equation

$$\zeta_{Nf}^{[']}(k) = -\left(\frac{1}{2}\omega_n\right)^2 2\int_k^\infty dk''\ \ F_N(k'')\left|\zeta_{Nf}^{[']}(k'')\right|^{-1} \tag{54}$$

for the determination of the collision coefficient $\zeta_{Nf}^{[']}$. Here we have assumed that $\zeta_{Nf}^{[']}$ and $\zeta_{Nf}^{['']}$ have the same spectral structure except for the difference of arguments k and k'' · The integral equation is solved to give

$$\zeta_{Nf}^{[']} = -\frac{1}{\sqrt{2}}\,\omega_n\,N_k^{['']}\ . \tag{55a}$$

With this solution we calculate all the expressions in (50) and (51) to obtain:

$$\mathcal{T}_{Xf}^{[']} = \frac{1}{3} c_X \left(\frac{1}{\sqrt{2}} \omega_n\right)^{-5} X_k^{[']2} \left\{ \frac{k''^2}{4} N_k^{['']-5} \right\} , \tag{55b}$$

$$K_N' = \left(\frac{1}{\sqrt{2}} \omega_n\right)^{-1} N_k^{[']2} \left\{ N_k^{['']-1} \right\} \tag{56a}$$

$$K_X' = \frac{1}{3} \left(\frac{1}{\sqrt{2}} \omega_n\right)^{-1} X_k^{[']2} \left\{ N_k^{['']-1} \right\} . \tag{56b}$$

6.4. Coupling process

The coupling function $\dot{W}^{[o]}$, as defined by (30), couples the two macro-fluctuations $\underset{\sim}{E}^{\circ}$ and $\underset{\sim}{X}^{\circ}$. Its determination requires a fluid equation of macro-evolution in the form

$$(\partial_t - i \nu_n \nabla^2 + i \tfrac{1}{2} \omega_n A^{\circ} N^{\circ} - \mathcal{T}_{Nf}) \underset{\sim}{E}^{\circ} = - i \underset{\sim}{X}^{\circ} , \tag{57}$$

as obtained by (25) under the hypothesis of fluidization and by using the asymptotic value of the collision coefficient at large scales.

Upon integrating (57), we get the macro-field

$$\underset{\sim}{E}^{\circ} = - i \int_0^{\infty} d\tau \, \Lambda^{\circ}(t,t-\tau) \, \underset{\sim}{X}^{\circ}(t-\tau) , \tag{58 a}$$

and calculate the coupling function

$$W^{[o]} = \int_0^{t\to\infty} d\tau \left\langle \underset{\sim}{X}^{o}(t) \cdot \Lambda^{o}(t, t-\tau)\ \underset{\sim}{X}^{o}(t-\tau) \right\rangle$$

$$= 3\ K_X^{[o]}\ . \tag{58b}$$

Here $K_X^{[o]}$ is the trace of the diffusivity tensor, and

$$\Lambda^{o} \simeq \Lambda^{[o]} \tag{59}$$

is the evolution operator. An approximate estimate is given in the following. In the Fourier form, (58 b) is

$$W^{[o]} = X_k^{[o]\,2} \left\{ \int_0^t d\tau\ h_N(\tau) \right\} . \tag{60}$$

In the fluid approximation, we have

$$h_{Nf} \cong \exp \zeta_{Nf}\ \tau\ , \tag{61}$$

and obtain

$$W^{[o]} = \left| \zeta_{Nf} \right|^{-1}\ X_k^{[o]\,2}\ . \tag{62}$$

7. Spectral Structure

7.1. Equation of spectral balance

We have obtained above the transfer function $T_X^{[0]}$ for the reverse cascade (52b) with the coefficient of collision $\zeta_{Xf}^{[1]}$ by (51b), the coupling function $W^{[0]}$ by (62), and finally the transfer function $T_N^{[0]}$ for the direct cascade (52a), with the coefficient $\zeta_{Nf}^{[1]}$ by (55a).

The three transport functions govern the spectral balance, as follows:

$$\varepsilon_E = -T_X^{[0]} + W^{[0]} - T_N^{[0]}$$

$$= \zeta_{Xf}^{[1]} E_k^{[0]2} + \left|\zeta_{Nf}\right|^{-1} X_k^{[0]2} + \zeta_{Nf}^{[1]} E_k^{[0]2} . \qquad (63)$$

The function

$$\varepsilon_E = \frac{1}{2} \partial_t \left\langle \left| \underset{\sim}{E}^0 \right|^2 \right\rangle$$

represents the instability of solitons.

The inertia subrange by reverse cascade is governed by

$$\varepsilon_E = \zeta_{Xf}^{[1]} E_k^{[0]2} . \qquad (64)$$

The coupling subrange at larger wavenumbers is governed by the spectral balance in the form:

$$W^{[o]} - T_N^{[o]} = 0 \quad , \tag{65a}$$

or

$$|\tau_{Nf}|^{-1} \; X_k^{[o]2} - |\tau_{Nf}| E_k^{[o]2} = 0 \quad . \tag{65b}$$

The asymptotic value of the collision coefficient is used in the present subrange of larger wavenumbers. The amount of energy $W^{[o]}$ that is produced by emission flows into the direct cascade $T_N^{[o]}$.

Formally, a constant $\dot{T}_N^{[o]}$ would predict a secondary inertia subrange by direct cascade, if a constant dissipation could be found. Even so, the spectral cutoff by $\iota\nu_n$ can occur before the direct cascade can be established.

7.2. Spectral laws

We take the spectral law

$$X_k^{[o]2} = \omega_X^{6} \quad k^{-2} \tag{66}$$

for the driving force, where ω_X is the frequency scale . This spectrum is valid for a driving force that serves as a production of internal density waves in compressible turbulence.[3]

The spectral structure in the inertia subrange by reverse cascade is governed by the spectral balance (64), under the conditions

(2) and (66). We find the spectral laws

$$E_k^{\,2} = c_E(\varepsilon_E^{-1}\,\omega_X^{\,6}\,\omega_n^{-5}\,\lambda^{-5})^{1/4} \quad , \qquad c_E = \left(\tfrac{1}{3}\tfrac{1}{4}\,i^{5/2}\,c_X\right)^{1/4} \tag{67a}$$

$$N_k^{\,2} = c_N(\varepsilon_E^{-1}\omega_X^{\,6}\,\omega_n^{-5}\,\lambda^{-1})^{1/2} \quad , \qquad c_N = c_E^{\,2} \quad , \tag{67b}$$

representing a flat maximum in the spectral plot.

The spectral structure in the coupling subrange is governed by the spectral balance (65b). By again using (2) and (66), we find the spectral laws:

$$E_k^{\,2} = |\zeta_{Nf}|^{-2}\,\omega_X^{\,6}\quad k^{-2} \tag{68a}$$

$$N_k^{\,2} = (\lambda\,|\zeta_{Nf}|^{-2}\omega_X^{\,6})^2\;k^{-4} \quad . \tag{68b}$$

Here $\zeta_{Nf} = \zeta_{Nf}^{[\cdot]}\Big|_{k=0}$ is an internal parameter defined by (55a). The spectral laws of intensities have their group notations omitted.

8. Conclusions

Although the equivalence between the Schrödinger equation and the Navier-Stokes equations has been established,[2, 4-6] their nonlinearlity differs. The former is cubically nonlinear and the latter has a nonlinearity of the gradient type. The Schrödinger equation is thus more suitable for the description of large-scale motions than do the Navier-Stokes equations.

The nonlineqr Schrödinger equation and the more general Zakharov equations were originally derived for weak fluctuations in plasmas.[1] Their derivations neglected the scattering (emission) by velocity fluctuations. By generalizing to include finite fluctuations, we found the forced Schrödinger equation.[2]

By the reverse cascade, the field energy is accumulated toward the large-scale end of the spectrum. The inertia subrange is followed by the coupling subrange, where the energy that is built up by emission is cascaded down toward the smaller eddies in a direct cascade. Consequently, the spectral intensity of field fluctuations falls with the power law k^{-2} , and the spectral intensity of density fluctuations falls with the power law k^{-4} .

The role of $i\nu_n$ is to cause a spectral cutoff of these power laws. The prediction of the power laws for the density intensity has been measured and is verified by Truc in the plasma experiments.[7] See Fig. 1. The plot uses the spectrol density $F_N(k)$ that is related to the spectral intensity by

$$N_k^{[1]\,2} = 2 \int_k^{\infty} dk'' \, F(k'') \quad . \tag{69}$$

The absence of the driving force from strong fluctuations will miss the mechanisms of accumulation and coupling. The transfer by direct cascade is the only mechanism which survives. On its own it cannot initiate a spectral flow and a spectral balance. This explains why it is so difficult for the standard Schrödinger equation to yield a spectrum.[8]

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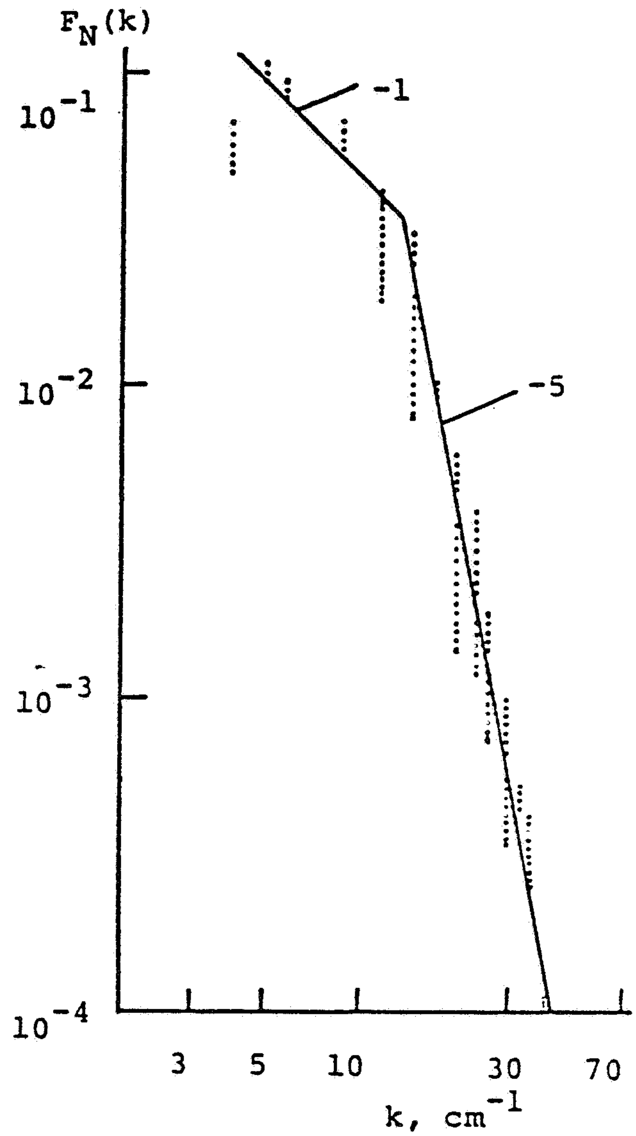

Fig. 1. Spectral results in plasma experiments by Truc.[7]

FLUID ANALOGUE OF THE SOLITON FORMALISM

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ABSTRACT

In view of the many applications to turbulent phenomena in fluids, plasmas, optics, astrophysics and nerve systems, a large effort has been given to investigate the solitons analytically and numerically. The results of analysis by means of the Z-S equations (the nonlinear Zakharov equations and the Schrödinger equation) have not been encouraging, and have concluded that the solitons could not become turbulent and establish a broadened spectrum. An examination of the mathematical foundation of these nonlinear equations reveals that they were indeed intended for weak fluctuations and not for turbulence. By a nonlinear analysis that carries all the nonlinear fluctuations including those from velocities, we develop a general soliton formalism that emphasizes the dynamics of the two-scale motions for a parallel development of turbulence in fluids and plasmas. From the Navier-Stokes equations for plasmas and compressible fluids of two scales (fast and slow waves), we derive two equations of propagation of density waves. The fast wave is related to the fast field by the property of the spontaneous creation of the field by rarefaction, and the slow density wave is related to the field intensity by the property

of the ponderomotive force. These two properties, which have been taken for granted in plasmas in the Z-S equations, are not evident in fluids. We demonstrate them by a Lagrangien formalism and a kinetic method as consistent with the Navier-Stokes dynamics. The first of the two properties transforms the hyperbolic equation of propagation of the fast density wave into a parabolic equation of evolution for the field-envelope as driven by the sound emission from finite velocity fluctuations. The second property gives the modulation. The emission and the modulation produce two nonlinearities. Our soliton system takes the generalized form of the Z-S equations by including the sound emission. It can be degenerated into the original Z-S equations by the omission of the sound emission in the weak fluctuation approximation. In our general formalism the solitons can be unstable and become turbulent, as occurring in the atmosphere. There, the measurments have shown that the solitons play a particularly important role in large scales, because the Navier-Stokes equations which are nonlinear in the form a gradient are not suitable.

1. Introduction

There has been a great interest recently in the modulational instability and turbulence of solitons, as described by the Schrödinger equation [1]

$$\left(i\partial_t + \nu_n \nabla^2 - \tfrac{1}{2}\omega_n \tilde{A}\tilde{N}\right)\tilde{\underline{E}}_a = 0 , \qquad (1)$$

with

$$\tilde{N} = -\lambda \left|\tilde{\underline{E}}_a\right|^2 , \qquad (2)$$

where $\tilde{\underline{E}}_a$ is the envelope of the field fluctuations, $\tilde{N}$ is a density fluctuation, $\tilde{A}$ is an operator of fluctuation, and

$$\nu_n , \quad \omega_n , \quad \text{and} \quad \lambda$$

are constant quantities having the dimensions of viscosity, frequency and $(\text{acceleration})^{-2}$.

The parabolic equation with cubic nonlinearity (1) has applications to a broad range of problems in nonlinear phenomena as connected with nonlinear optics, plasma, nerve systems, internal gravity waves, water waves and fluid dynamics. It has been proposed already in 1944 by Leontovich [2] for analysing the nonlinear propagation of light and radio waves. The soliton description has become one of the standard method for treating nonlinear waves in multi-scale systems, where the fast waves are modulated nonlinearly by slow waves. Its importance to instability of plasmas and to various problems of solid state has been demonstated by the large amount of analytical and numerical works in the current literature. Its relevance to gravity wave and water waves [3,4], as well as to electronic excitations in long helical molecules [5] has also been demonstrated.

The soliton description has been shown to be valid in a wide variety of other applications in fluids .rotating flows, acoustic turbulence, the dynamics of liquid sheets, thermal convective instability, Rayleigh instability, and instability of Poiseuille flow. In the atmosphere, many of these instabilities will develop turbulence.

The importance of the cubic nonlinearity to fluid dynamical turbulence has been pointed out already in 1944 by Landau [6], and the cubic parabolic equation in the real form has been proposed by Stewartson and Stuart [7].

The equivalence between the Navier-Stokes equations and the soliton formalism has been shown by several authors for fluids and plasmas. The equivalence in fluid [8-10] was based upon the Madelung transformation

$$|\underset{\sim}{\tilde{E}}| \sim (|\tilde{N}|)^{\frac{1}{2}} \qquad (3)$$

The derivation was formal, involving certain restrictive conditions (irrotational flow and special barotropic property), and did not distinguish between scales. On the other hand, the equivalence in plasmas [1] did distinguish scales, but took the properties of space charge and ponderomotive force as axioms, without going into their mathematical foundation [11,12]. The Schrödinger equation thus derived suffers from its basis from weak fluctuations.

In the following, we develop a nonlinear theory of solitons to be valid for both fluid and plasma. The nonlinearities come from two sources : The modulational nonlinearity is based upon the ponderomotive force from the large-scale (or low frequency) density fluctuations, and the nonlinear emission of sound is produced by the strong velocity fluctuations of small-scales (or high frequencies). The density fluctuations of two scales act differently on the field : The density fluctuation of small-scale creates a spontaneous field-divergence, known as the space charge in plasma, and the density fluctuation of large-scale responds to the ponderomotive force, that is a known property in plasma and enters as the Madelung hypothesis (3) in fluid. We develop two theories to explain these two phenomena.

2. Hydrodynamics of multi-scale motions.

2.1. Scaling into slow and fast motions

A fluctuating quantity ($\hat{}$) can be decomposed into an ensemble average ($\bar{}$) by the operator $\bar{A}$ and a fluctuation by the operator $1 - \bar{A}$, where 1 is the unit operator. In multi-scale problems, we distinguish between a slow fluctuation ($\tilde{}$) and a fast fluctuation ($\approx$) by the scaling operators

$$\tilde{A} \text{ and } \overset{\approx}{A} \quad . \tag{4a}$$

These operators add to

$$1 = \bar{A} + \tilde{A} + \overset{\approx}{A}, \tag{4b}$$

and correspondingly, a fluctuating quantity has the following components

$$\hat{(\,)} = \overline{(\,)} + \tilde{(\,)} + \overset{\approx}{(\,)} \quad . \tag{4c}$$

A fast fluctuation

$$\overset{\approx}{(\,)} = \frac{1}{2}\tilde{(\,)}_a \left[e^{-(i\omega_n + \varepsilon)t} + c\,c \right] \tag{5}$$

has an amplitude, or envelope, that varies slowly. It is denoted by $(\tilde{\ })_a$ and is obtained by the scaling operator $\tilde{A}_a$. The representative frequency is ω_n. An infinitesimal rate of damping ε written for convergence will be omitted eventually. The complex conjugate part is denoted by c.c.

The velocity $\hat{\underline{u}}$ in compressible fluid consists of a transverse mode $\hat{\underline{u}}^t$ and a longitudinal mode $\hat{\underline{u}}^\ell$.

2.2. Basic equations

We consider the equations of continuity and momentum as follows .

$$\partial_t \hat{n} + \nabla_j \hat{n} \hat{u}_j = 0 \qquad (6)$$

$$\partial_t \hat{n} \hat{u}_i + \nabla_j \hat{n} \hat{u}_j \hat{u}_i = -c^2 \nabla_i \hat{n} + \sigma \hat{n} \hat{E}_i \ . \qquad (7)$$

The longitudinal mode is represented by the collective field $\hat{E}_i$. The fluid velocity

$$\hat{\underline{u}} = \hat{\underline{u}}^t + (1-\sigma) \hat{\underline{u}}^\ell \qquad (8)$$

may include both the transverse mode $\hat{\underset{\sim}{u}}^t$ and the longitudinal mode $\hat{\underset{\sim}{u}}^\ell$ by putting $\sigma = 0$, or may represent the transverse mode only by putting $\sigma = 1$. The speed of sound is c in isothermal fluid, so that the pressure gradient is $c^2 \nabla \hat{n}$. The density is normalized to unity as

$$\bar{A}\hat{n} = 1 \tag{9}$$

We consider a homogeneous medium with .

$$\bar{A}\hat{\underset{\sim}{u}} = 0, \qquad \bar{A}\hat{\underset{\sim}{E}} = 0. \tag{10}$$

It will be conveniant to transform (6) and (7) into the wave form by cross differentiation for eliminating the common term $\nabla_j \partial_t \hat{n}\hat{u}_j$, yielding the equation of propagation

$$(\partial_t^2 - c^2\nabla^2)\hat{n} + \sigma\nabla_j \hat{n}\hat{E}_j = \hat{r}^t + (1-\sigma)\hat{r}^\ell, \tag{11}$$

where

$$\hat{r} \equiv \underset{\sim}{\nabla}\underset{\sim}{\nabla} \cdot \hat{n}\hat{\underset{\sim}{u}}\hat{\underset{\sim}{u}} \tag{12}$$

is the scattering (or emission) function.

Since the two forms of the wave equation (11), either with $\sigma = 1$ or with $\sigma = 0$, are equivalent, we have the condition

$$-\underset{\sim}{\nabla}\,\hat{n}\,\underset{\sim}{\hat{E}} = \hat{r}^{\ell}, \qquad (13)$$

defining the collective field $\hat{E}_j$. The product

$$\hat{n}\,\underset{\sim}{\hat{E}} \qquad (14)$$

is called the modulation.

The scattering function

$$\begin{aligned}\hat{r} &\equiv \underset{\sim}{\nabla}\underset{\sim}{\nabla} : \hat{n}\,\underset{\sim}{\hat{u}}\,\underset{\sim}{\hat{u}} \\ &= \hat{r}^{t} + \hat{r}^{\ell}\end{aligned} \qquad (15)$$

has two modes : the transverse mode is

$$\hat{r}^{t} \equiv \underset{\sim}{\nabla}\underset{\sim}{\nabla} \cdot \left(\hat{n}\,\underset{\sim}{\hat{u}}\,\underset{\sim}{\hat{u}}\right)^{t} \qquad (16a)$$

and the longitudinal mode is

$$\hat{r}^{\ell} \equiv \underset{\sim}{\nabla}\underset{\sim}{\nabla} : (\hat{n}\, \hat{\underset{\sim}{u}}\, \hat{\underset{\sim}{u}})^{\ell} , \tag{16b}$$

as denoted by the superscripts

$$(\ )^{t} \quad \text{and} \quad (\ )^{\ell} , \tag{17}$$

respectively.

By the second derivative $\underset{\sim}{\nabla}\underset{\sim}{\nabla}$, the scattering function is controlled by small scales, so that we can approximate the total scattering by

$$\hat{r} \cong \underset{\sim}{\nabla}\underset{\sim}{\nabla} : \hat{\underset{\sim}{u}}\, \hat{\underset{\sim}{u}} . \tag{18}$$

The longitudinal component has a fast fluctuation

$$\tilde{\tilde{r}}^{\ell} \equiv \underset{\sim}{\nabla}\underset{\sim}{\nabla} \cdot \tilde{\tilde{\underset{\sim}{u}}}^{\ell}\, \tilde{\tilde{\underset{\sim}{u}}}^{\ell} \tag{19}$$

with the amplitude

$$\tilde{r}_a^{\ell} \equiv \nabla\nabla : \; \hat{\underline{u}}_a^{\ell} \; \tilde{\underline{u}}_a^{\ell *} \tag{20}$$

and a slow fluctuation

$$\tilde{r}_N \equiv \underline{\nabla}\underline{\nabla} : \; \tilde{\underline{u}}_N^{\ell} \; \tilde{\underline{u}}_N^{\ell} \; . \tag{21}$$

The index $(\;)_N$ denotes the slow fluctuation as distinct from $(\;)_a$. We assume that the correlations between different modes are negligible. Analogous definitions can be written for the transverse mode. Such an approximation is usual in sound propagation in a turbulent medium. The quantities

$$\hat{n}, \; \hat{\underline{u}}, \; \hat{\underline{E}}, \; \hat{r} \tag{22}$$

with $(\hat{\;})$ refer to both the rapidly fluctuating motion and the slowly fluctuating motion , as distinguished by the notations

$$n, \; \underline{u}, \; \underline{E}, \; r, \; \omega_n, \; c_n \tag{23a}$$

and

$$N, \; \underline{u}_N, \; \underline{E}_N, \; r_N, \; \omega_N, \; c_N \tag{23b}$$

with the characteristic frequencies

$$\omega_n \quad \text{and} \quad \omega_N \ , \tag{24}$$

for the wave motions of high and low frequency scales, respectively. These two-scaled motions may be called the two species, indicated by a subscript $(\ )_\beta$.

When the two scales (23a) and (23b) are treated as the two species, it might be convenient for a certain purpose to write the fields

$$\tilde{\underset{\sim}{E}} = \hat{e}_n \, \zeta_n \, \tilde{\underset{\sim}{\mathbb{E}}} \tag{25a}$$

$$\tilde{\underset{\sim}{E}}_N \equiv \hat{e}_N \, \zeta_N \, \tilde{\underset{\sim}{\mathbb{E}}} \tag{25b}$$

in terms of a common field $\tilde{\underset{\sim}{\mathbb{E}}}$, to be distinguished by the parameters ζ_n and ζ_N in the ratio

$$\xi \equiv \zeta_n / \zeta_N = (\omega_n / \omega_N)^2 \gg 1 \tag{26}$$

The coefficient

$$\hat{e}_\beta = \pm 1 \tag{27}$$

has a $+$ sign for a neutral fluid, and $\hat{e}_N = 1$, $\hat{e}_n = -1$ for two species of opposite charges. The parameters depend on the species considered, e.g. on masses and charges in plasmas.

3. Hyperbolic and parabolic equations for the propagation of density waves.

3.1. Notations

For the fast species, we define

$$\hat{n} = 1 + \tilde{n} + \tilde{\tilde{n}}, \quad \hat{\underset{\sim}{E}} = \tilde{\underset{\sim}{E}} + \tilde{\tilde{\underset{\sim}{E}}}, \quad \text{with} \quad \bar{\underset{\sim}{E}} = 0, \tag{28a}$$

and

$$\tilde{A}\,\hat{n} = \tilde{n}, \qquad \tilde{\tilde{A}}\,\hat{n} = \tilde{\tilde{n}} \tag{28b}$$

$$\tilde{A}\,\hat{\underset{\sim}{E}} = \tilde{\underset{\sim}{E}}, \qquad \tilde{\tilde{A}}\,\hat{\underset{\sim}{E}} = \tilde{\tilde{\underset{\sim}{E}}}, \tag{28c}$$

and scale the modulation as follows :

$$\tilde{A}\,\hat{n}\,\hat{\underset{\sim}{E}} = \tilde{A}(1+\tilde{n})\tilde{\underset{\sim}{E}} + \tilde{A}\,\tilde{\tilde{n}}\,\tilde{\tilde{\underset{\sim}{E}}} \tag{29a}$$

$$\begin{aligned} \tilde{\tilde{A}}\,\hat{n}\,\hat{\underset{\sim}{E}} &= (1+\tilde{n})\tilde{\tilde{\underset{\sim}{E}}} + \tilde{\tilde{n}}\,\tilde{\underset{\sim}{E}} + \tilde{\tilde{A}}\,\tilde{\tilde{n}}\,\tilde{\tilde{\underset{\sim}{E}}} \\ &\simeq (1+\tilde{n})\tilde{\tilde{\underset{\sim}{E}}} + \tilde{\tilde{A}}\,\tilde{\tilde{n}}\,\tilde{\tilde{\underset{\sim}{E}}}, \end{aligned} \tag{29b}$$

$$\tilde{A}\,\tilde{n}\,\tilde{\underset{\sim}{E}} = \tilde{n}\,\tilde{E} - \langle \tilde{n}\,\tilde{E} \rangle$$
$$\cong \tilde{n}\,\tilde{\underset{\sim}{E}} \qquad (29c)$$

$$\tilde{\tilde{A}}\,\tilde{\tilde{n}}\,\tilde{\tilde{\underset{\sim}{E}}} = \tilde{\tilde{n}}\,\tilde{\tilde{\underset{\sim}{E}}} - (\bar{A} + \tilde{A})\,\tilde{\tilde{n}}\,\tilde{\tilde{\underset{\sim}{E}}}$$
$$= \tilde{\tilde{n}}\,\tilde{\tilde{\underset{\sim}{E}}} - \tilde{A}\,\tilde{\tilde{n}}\,\tilde{\tilde{\underset{\sim}{E}}}\,, \qquad (29d)$$

in view of

$\bar{A} + \tilde{A} = 1$, for a slowly fluctuating function (30a)

$\bar{A} + \tilde{A} + \tilde{\tilde{A}} = 1$, for a rapidly fluctuating function (30b)

and

$\bar{A}(\ \ ) = 0$, in a homogeneous medium. (30c)

Here 1 is the unit operator.

3.2. Hyperbolic equation for the propagation of high-frequency density waves

By applying the scaling operator $\tilde{\tilde{A}}$, we transform (11) with $\sigma = 1$ into :

$$\left(\partial_t^2 - c^2\nabla^2\right)\tilde{\tilde{n}} + \nabla_j \tilde{\tilde{A}}\,\hat{n}\,\hat{E} = \tilde{\tilde{\mathcal{R}}}^{t}. \qquad (31)$$

The modulation takes the approximate form

$$\tilde{\tilde{A}}\,\hat{n}\,\hat{\underset{\sim}{E}} \cong (1 + \tilde{N})\,\tilde{\tilde{\underset{\sim}{E}}} + \tilde{\tilde{A}}\,\tilde{\tilde{n}}\,\breve{\tilde{\tilde{\underset{\sim}{E}}}}\ . \qquad (32)$$

from (29b), upon replacing $\tilde{n}$ by $\tilde{N}$ for coupling between the two scales of density waves.

Upon substituting (32) into (31) we derive the wave equation for the fast mode, as follows :

$$\left(\partial_t^2 - c^2\nabla^2\right)\tilde{\tilde{n}} + \nabla_j\left(1+\tilde{N}\right)\tilde{\tilde{E}}_j = \tilde{\tilde{\lambda}} \tag{33}$$

The driving force is

$$\begin{aligned} \tilde{\tilde{\lambda}} &= \tilde{\tilde{A}}\hat{\lambda}^t - \nabla_j\left(\tilde{\tilde{A}}\,\tilde{\tilde{n}}\,\tilde{\tilde{E}}_j + \tilde{\tilde{n}}\tilde{E}_j\right) \\ &\cong \tilde{\tilde{A}}\hat{\lambda}^t - \nabla_j \tilde{\tilde{A}}\,\tilde{\tilde{n}}\,\tilde{\tilde{E}}_j \\ &\equiv \tilde{\tilde{A}}\left(\hat{\lambda}^t + \hat{\lambda}^\ell\right). \end{aligned} \tag{34}$$

There we have neglected the compling $\tilde{\tilde{n}}\,\tilde{E}_j$ of different scales and have made use of the definition (13).

3.3. Parabolic equation for the envelope of density waves

The envelope of the fast wave has been defined by (5) and is obtained by the operator $\tilde{A}_a$. Upon applying this operator and neglecting the second time derivative $\partial_t^2\,\tilde{n}_a$ of the slowly fluctuating envelope, we transform the hyperbolic equation (33) into the parabolic equation :

$$\left(i\partial_t + \nu_n \nabla^2\right)\tilde{n}_a - \frac{1}{2\omega_n}\nabla \tilde{N}\tilde{E}_a = -\frac{1}{2\omega_n}\left(\tilde{r}_a + \tilde{\theta}_a\right), \tag{35}$$

with

$$\nu_n \equiv c^2/2\omega_n \tag{36}$$

and

$$\tilde{\theta}_a \equiv \omega_n^2 \tilde{n}_a - \nabla \tilde{E}_a \tag{37}$$

3.4. Hyperbolic equation for the propagation of low frequency density waves

In an analogous manner and by applying $\tilde{A}$ to (11), we obtain the wave equation for the slow species of frequency scale ω_N, as follows

$$\left(\partial_t^2 - c_N^2 \nabla^2\right)\tilde{N} = \tilde{Q}_M + \tilde{Q}_r - \nabla\cdot \tilde{E}_N\ , \tag{38}$$

with

$$\tilde{Q}_M \equiv -\nabla_j \tilde{N}\tilde{E}_N = \tilde{r}_N^{\ell}\ , \tag{39a}$$

$$\tilde{Q}_n \equiv \tilde{A}\,\tilde{n}_N^{\,t} + \tilde{A}\,\tilde{\tilde{n}}_N^{\,\ell} \cong \tilde{n}_N^{\,t} , \tag{39b}$$

by (13).

On the other hand, the species of fast scale ω_n may have a density $\tilde{n}$ that varies slowly according to the wave equation

$$\begin{aligned}(\partial_t^2 - c^2\nabla^2)\,\tilde{n} &= \tilde{n}^{\ell} + \tilde{n}^{t} - \tilde{A}\,\nabla_j \tilde{\tilde{n}}\,\tilde{\tilde{E}}_j - \underline{\nabla}\,\underline{\tilde{E}} \\ &= \tilde{n}^{\ell} + \tilde{n}^{t} - \tilde{A}\,\nabla_j \tilde{\tilde{n}}\,\tilde{\tilde{E}}_j - \hat{e}_n \xi\,\underline{\nabla}\,\underline{\tilde{E}}_N ,\end{aligned} \tag{40}$$

where we have written

$$\tilde{n}^{\ell} = -\tilde{A}\,\nabla_j \tilde{n}\,\tilde{E}_j \tag{41a}$$

and

$$\underline{\tilde{E}} = \hat{e}_n\,\xi\,\underline{\tilde{E}}_N , \tag{41b}$$

from (13), (25) and (26).

Upon dividing (40) by ξ, we have

$$\xi^{-1}\left(\partial_t^2 - c^2\nabla^2\right)\tilde{n} = \tilde{Q}_M + \tilde{Q}_r - \hat{e}_n \nabla \cdot \tilde{E}_N , \tag{42}$$

with

$$\tilde{Q}_M \equiv \xi^{-1}\tilde{r}^{\ell} \tag{43a}$$

$$\tilde{Q}_r \equiv \xi^{-1}\left(\tilde{r}^t - \tilde{A}\nabla_j \tilde{\tilde{n}}\tilde{\tilde{E}}_j\right). \tag{43b}$$

By combining the wave equations (38) and (42) for the two slowly varying densities $\tilde{N}$ and $\tilde{n} \doteq \tilde{N}$, respectively, we obtain

$$\left(\partial_t^2 - c_s^2\nabla^2\right)\tilde{N} = 2\left(\tilde{Q}_M + \tilde{Q}_r\right) - \left(1+\hat{e}_n\right)\nabla \cdot \tilde{E}_N \tag{44}$$

with

$$c_s^2 = c_N^2 + \xi^{-1} c^2 \tag{45}$$

Here we have neglected the term $\xi^{-1}\partial_t^2\tilde{n}$, by $\xi > 0$, from (26).

The term $(1 + \hat{e}_n)\, \underline{\nabla} \cdot \underline{\tilde{E}}_N$ is canceled in plasma by $\hat{e}_n = -1$, but it survives in fluids by $\hat{e}_n = +1$, from (27).

4. Theory of ponderomotive force for plasmas and fluids

The dynamical equation

$$\frac{d\hat{\underset{\sim}{u}}^{l}}{dt} = \hat{\underset{\sim}{E}} \quad , \tag{46}$$

with

$$\frac{d}{dt} \equiv \partial_t + \hat{\underset{\sim}{u}} \cdot \underset{\sim}{\nabla} \quad ,$$

describes the evolution of the longitudinal velocity under the collective field. It can be integrated to get

$$\hat{\underset{\sim}{u}}^{l} = \int_0^t d\tau' \; \hat{\underset{\sim}{E}}(t-\tau') \tag{47}$$

in the Lagrangian description. The Lagrangian field

$$\hat{\underset{\sim}{E}}(t-\tau') = \hat{E}\left[t-\tau', \hat{\underset{\sim}{x}}(t-\tau')\right] \tag{48}$$

is defined as the field at the time $t - \tau$ along the trajectory.

We can calculate the product

$$\hat{\underset{\sim}{u}}^{l} \hat{\underset{\sim}{u}}^{l} = \int_0^t d\tau' \int_0^t d\tau'' \; \hat{\underset{\sim}{E}}(t-\tau') \hat{\underset{\sim}{E}}(t-\tau'') \,. \tag{49}$$

The Fourier formula

$$\hat{\underset{\sim}{E}}(t-\tau') = \int d\omega' \, e^{i\omega'(t-\tau')} \hat{\underset{\sim}{E}}(\omega') \tag{50}$$

gives the integrand

$$\begin{aligned} \hat{\underset{\sim}{E}}(t-\tau')\hat{\underset{\sim}{E}}(t-\tau'') = \iint d\omega' d\omega'' \, & e^{i(\omega'+\omega'')t} \\ \times \, & e^{-i(\omega'\tau'+\omega''\tau'')} \\ \times \, & \hat{\underset{\sim}{E}}(\omega')\hat{\underset{\sim}{E}}(\omega''). \end{aligned} \tag{51}$$

In a homogeneous medium, we can take the average with respect to t in the interval of time $2T$. It is assumed that the scale of t is larger than any characteristic time scales. We find

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} dt \; e^{i(\omega'+\omega'')t} = \chi\,\delta(\omega'+\omega'') \tag{52}$$

The factor

$$\chi \equiv \pi/T$$

is called the factor of Fourier truncation. After the time average, (51) is reduced into

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} dt\ \hat{\underset{\sim}{E}}(t-\tau')\hat{\underset{\sim}{E}}(t-\tau'') = \int d\omega'\ e^{-\iota\omega'(\tau'-\tau'')}$$

$$\times\ \cancel{\chi}\ \hat{\underset{\sim}{E}}(\omega')\hat{\underset{\sim}{E}}(-\omega')$$

$$= \hat{\underset{\sim}{E}}\hat{\underset{\sim}{E}}\ e^{-\iota\omega_n(\tau'-\tau'')}\ e^{-\varepsilon(\tau'+\tau'')}, \qquad (53)$$

if the high-frequency field is peaked at ω_n as its characteristic frequency, by (24), (5), and by writing

$$\cancel{\chi}\ \hat{\underset{\sim}{E}}(\omega')\ \hat{\underset{\sim}{E}}(-\omega') = \hat{\underset{\sim}{E}}\hat{\underset{\sim}{E}}\ \delta(\omega-\omega_n) \qquad (54)$$

Upon substituting (53) into (49), we obtain

$$\hat{\underset{\sim}{u}}^{l}\hat{\underset{\sim}{u}}^{l} = \hat{\underset{\sim}{E}}\hat{\underset{\sim}{E}}\ |\tau_n|^2 \qquad (54a)$$

with

$$|\tau_n|^2 \equiv \int_0^{t\to\infty} d\tau' \int_0^{t\to\infty} d\tau''\ e^{-\iota\omega_n(\tau'-\tau'')}\ e^{-\varepsilon(\tau'+\tau'')} \qquad (54b)$$

The infinitesimal damping ε guarantees the convergence. The upper limit t of integration can be put equal to ∞ without altering the value of the integral.

The double integral is now evaluated by writing

$$\tau_n \equiv \int_0^\infty d\tau' \, e^{-(i\omega_n + \varepsilon)\tau'}$$

$$= \frac{1}{i\omega_n + \varepsilon} \tag{55}$$

to find

$$|\tau_n|^2 \equiv \tau_n \tau_n^*$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\omega_n^2 + \varepsilon^2} = \omega_n^{-2} . \tag{56}$$

This yields the tensor

$$\hat{\underset{\sim}{u}}^{l} \, \hat{\underset{\sim}{u}}^{l} = \omega_n^{-2} \, \hat{\underset{\sim}{E}} \, \hat{\underset{\sim}{E}} \tag{57}$$

to calculate the scattering functions

$$\hat{r}^{l} \equiv \underset{\sim}{\nabla} \underset{\sim}{\nabla} : \hat{\underset{\sim}{u}}^{l} \, \hat{\underset{\sim}{u}}^{l}$$

$$= \omega_n^{-2} \, \underset{\sim}{\nabla} \underset{\sim}{\nabla} : \hat{\underset{\sim}{E}} \, \hat{\underset{\sim}{E}} \tag{58}$$

and

$$2\tilde{Q}_M = 2\xi^{-1} \omega_n^{-2} \, \tilde{A} \, \underset{\sim}{\nabla} \underset{\sim}{\nabla} : \tilde{\underset{\sim}{E}} \, \tilde{\underset{\sim}{E}}$$

$$= \xi^{-1} \omega_n^{-2} \, \tilde{A} \, \nabla_i \nabla_j \, |\tilde{\underset{\sim}{E}}_a|^2$$

$$= \nabla^2 \tilde{\mathcal{E}} \, . \tag{59}$$

by (18) and (43a), finding the ponderomotive potential

$$\tilde{\zeta} = d^{-1} \xi^{-1} \omega_n^{-2} \tilde{A} \left| \tilde{\underset{\sim}{E}}_a \right|^2 . \tag{60}$$

In (58b), we have written

$$\tilde{E}_i \tilde{E}_j = \frac{1}{d} \tilde{\underset{\sim}{E}} \tilde{\underset{\sim}{E}} \delta_{ij} . \tag{61a}$$

and

$$\tilde{\underset{\sim}{E}} \cdot \tilde{\underset{\sim}{E}} = \frac{1}{2} \left| \tilde{\underset{\sim}{E}}_a \right|^2 \tag{61b}$$

in d dimensions. The formula (60) for the ponderomotive force is derived for its general validity to plasmas and fluids.

From (44), (43) and (59), it is seen that the density wave $\tilde{N}$ is produced by an emission $\tilde{\underset{\sim}{Q}}$ by the transverse component of the velocity fluctuations, and a source $\tilde{Q}_M$ by the longitudinal component of the velocity fluctuations in the form of a ponderomotive force.

5. Spontaneous creation of the collective field by the density waves (a kinetic description)

As a starting basis, we use the master equation

$$(\partial_t + \hat{\mathcal{L}})\,\hat{F}(t, \underset{\sim}{x}, \underset{\sim}{v}) = 0 \tag{62}$$

for the distribution function

$$\hat{F}(t, \underset{\sim}{x}, \underset{\sim}{v}) = \hat{n}\,\delta[\underset{\sim}{v} - \underset{\sim}{\hat{u}}(t, \underset{\sim}{x})] \;, \tag{63}$$

with the differential operator

$$\hat{\mathcal{L}} = \underset{\sim}{v}\cdot\underset{\sim}{\nabla} + \underset{\sim}{\hat{E}}(t, x)\cdot\underset{\sim}{\partial} \;, \quad \underset{\sim}{\partial} \equiv \partial/\partial\underset{\sim}{v} \;. \tag{64}$$

The kinetic description (62) replaces the dynamical description (46). By considering the operator $\delta \equiv 1 - \bar{A}$ that gives the deviation from the average, we transform (62) into

$$(\partial_t + \delta\hat{\mathcal{L}})\,\delta F = -\,\delta\underset{\sim}{E}\cdot\underset{\sim}{\partial}\bar{F} \;. \tag{65}$$

By integrating, we obtain the distribution

$$\delta F = -\int_0^t d\tau\;\Lambda(t, t-\tau)\;\delta\underset{\sim}{E}(t-\tau)\cdot\underset{\sim}{\partial}\bar{F} \,, \tag{66}$$

and by taking the zeroth moment we find the density

$$\delta n(t,\underset{\sim}{x}) \equiv \int d\underset{\sim}{v}\ \delta F(t,\underset{\sim}{x},\underset{\sim}{v})$$

$$= -\int_0^t d\tau \int d\underset{\sim}{v}\ \Lambda(t,t-\tau)\ \delta \underset{\sim}{E}(t-\tau)\ \underset{\sim}{\partial} \bar{F}\ . \tag{67}$$

The evolution operator Λ governs the Lagrangian description of the field $\delta E\ (t - \tau)$ along the phase trajectory, such that

$$\Lambda(t,t-\tau)\ \delta\underset{\sim}{E}(t-\tau) = \delta\underset{\sim}{E}\left[t-\tau,\ \underset{\sim}{\hat{x}}(t-\tau)\right] \tag{68}$$

The position

$$\underset{\sim}{\hat{x}}(t-\tau) = \underset{\sim}{x} - \underset{\sim}{v}\tau - \underset{\sim}{\hat{\ell}}(\tau) \tag{69}$$

along the trajectory consists of a portion $\underset{\sim}{x} - \underset{\sim}{v}\tau$ from the streaming by $\underset{\sim}{v}$ in the Eulerian description, and another portion from the path $\hat{\ell}$ as depending on the field in the Lagrangian description.

By the use of the Fourier expansion, the Lagrangian field (68) gives

$$\Lambda(t,t-\tau)\ \delta\underset{\sim}{E}\ \underset{\sim}{\partial} \equiv \delta\underset{\sim}{E}\left[t-\tau,\ \underset{\sim}{\hat{x}}(t-\tau)\right]\cdot\underset{\sim}{\partial}$$

$$= \iint d\omega\, d\underset{\sim}{k}\ e^{i(\omega t - \underset{\sim}{k}\,\underset{\sim}{x})}\ \underset{\sim}{\partial}\cdot h(\tau)\ \delta\underset{\sim}{E}(\omega,\underset{\sim}{k}) \tag{70}$$

The orbit function

$$h(\tau) = h_v(\tau)\, h_E(\tau) \tag{71}$$

has the following two components, by (69) . The kinematic component of streaming

$$h_v(\tau) \equiv e^{-i(\omega - \underset{\sim}{k}\,\underset{\sim}{v})\tau} \tag{72a}$$

gives

$$\underset{\sim}{\partial} h_v(\tau) = i\underset{\sim}{k}\,\tau\, h_v(\tau) \quad . \tag{72b}$$

The dynamic component

$$\begin{aligned} h_{E\beta}(\tau) &\equiv e^{ik\cdot \hat{l}_\beta(\tau)} \\ &\cong e^{(i\,\omega_\beta - \epsilon)\tau} \end{aligned} \tag{73}$$

is characteristic of the oscillation and the scale of the field for the species β If the characteristic frequency ω_β dominates the dynamics of the oscillating field, the approximation is valid. The hypothesis that the Fourier component has a singular peak at ω_β without broadening is valid in our present derivation of soliton equation for the laminar flow, which subsequently may serve to describe the microdynamical state of turbulence.

Upon substituting (72b) and (73) into (71), and subsequently into (70), we obtain .

$$\Lambda(t,t-\tau)\,\Gamma \underset{\sim}{E}_\beta \cdot \underset{\sim}{\partial} = \iint d\omega\, d\underset{\sim}{k}\; e^{\iota(\omega t - \underset{\sim}{k}\,\underset{\sim}{x})}\, \tau\, \iota \underset{\sim}{k} \cdot \delta \underset{\sim}{E}_\beta(\omega,\underset{\sim}{k})$$

$$\times\, e^{(\iota\omega_\beta - \varepsilon)\tau} h_v(\tau)$$

$$\cong \tau e^{(\iota\omega_\beta - \varepsilon)\tau} \iint d\omega d\underset{\sim}{k}\; e^{\iota(\omega t - \underset{\sim}{k}\,\underset{\sim}{x})}\, \iota \underset{\sim}{k} \cdot \delta \underset{\sim}{E}_\beta(\omega,\underset{\sim}{k})$$

$$= \tau \left| e^{(\iota\omega_\beta - \varepsilon)\tau} \right| \underset{\sim}{\nabla} \cdot \delta \underset{\sim}{E}_\beta(t,\underset{\sim}{x}) \qquad (74)$$

The approximation of neglecting the streaming as compared to the dominant frequency has been made The characterization by ω_β specifies the field, the distribution, and the density, as

$$\delta \underset{\sim}{E}_\beta , \quad \delta F_\beta , \quad \delta n_\beta$$

From (67), (74), we calculate the distribution function

$$\delta F_\beta \cong -\int_0^{t\to\infty} d\tau\, \tau \left| e^{-(\iota\omega_\beta + \varepsilon)} \right| (\bar{F} + \tilde{F})\, \underset{\sim}{\nabla} \cdot \delta \underset{\sim}{E}_\beta(t,\underset{\sim}{x}), \qquad (75)$$

and the density

$$\delta n_\beta = -\underset{\sim}{\nabla} \cdot \delta \underset{\sim}{E}_\beta \int_0^{t\to\infty} d\tau\, \tau \left| e^{-(i\omega_\beta + \varepsilon)\tau} \right| \qquad (76)$$

by taking the moment. The integral calculated as

$$\lim_{\varepsilon \to 0} \int_0^{t \to \infty} d\tau \; \tau \left| e^{-(i\omega_\beta + \varepsilon)\tau} \right| = \omega_n^{-2} , \tag{77}$$

reduces (76) into

$$\delta n_\beta = - \omega_\beta^{-2} \underset{\sim}{\nabla} \cdot \delta \underset{\sim}{E}_\beta \quad . \tag{78}$$

By a change of variables from the specified field $\delta \underset{\sim}{E}_\beta$ into the common field $\delta \underset{\sim}{E}$ by (25), we find

$$\underset{\sim}{\nabla} \cdot \delta \underset{\sim}{E} = - \gamma \; \hat{e}_\beta^{-1} \; \delta n_\beta \tag{79}$$

for one single species, or

$$\underset{\sim}{\nabla} \cdot \delta \underset{\sim}{E} = - \gamma \sum_\beta \hat{e}_\beta^{-1} \; \delta n_\beta \tag{80}$$

for many species. Here

$$\gamma \equiv \frac{\omega_\beta^2}{\zeta_\beta} = \frac{\omega_n^2}{\zeta_n} = \frac{\omega_N^2}{\zeta_N} \tag{81}$$

is a factor independent of the species, by (26).

For two species of densities $\hat{N}$ and $\hat{n}$ and characteristic frequencies ω_n and ω_N, (80) is

$$\underset{\sim}{\nabla} \cdot \delta \underset{\sim}{\mathbb{E}} = -\gamma\left(\hat{e}_n^{-1}\delta n + \hat{e}_N^{-1}\delta N\right). \tag{82}$$

In particular, we have

$$\underset{\sim}{\nabla}\, \underset{\sim}{\tilde{\tilde{\mathbb{E}}}} = -\gamma\, \hat{e}_n^{-1}\, \tilde{\tilde{n}}, \qquad \text{for the fast field,} \tag{83a}$$

and

$$\begin{aligned} \underset{\sim}{\nabla} \cdot \underset{\sim}{\tilde{\mathbb{E}}} &= -\gamma\left(\hat{e}_n^{-1}\tilde{n} + \hat{e}_N^{-1}\tilde{N}\right) \\ &= -\gamma\left(\hat{e}_n^{-1} + \hat{e}_N^{-1}\right)\tilde{N}, \qquad \text{for the slow field.} \end{aligned} \tag{83b}$$

The relations (83) describes the spontaneous creation of the self-consistent field by the density fluctuations without going through a process of evolution. These relations are valid for fluids with $\hat{e}_\beta = 1$ and for plasmas with $\hat{e}_n = -1$, $\hat{e}_N = +1$.

6. The forced Žakharov equations

6.1. Envelope equation for the field

In (35) we have derived the envelope equation for the density $\tilde{n}_a$. By the relationship (83a), rewritten as

$$\nabla \tilde{E}_a = -\omega_n^2 \tilde{n}_a , \tag{84}$$

by the use of (25a) and (81), the envelope equation for $\tilde{n}_a$ can be transformed into an envelope equation for $\tilde{E}_a$, with

$$\tilde{\theta}_a = 0 \tag{85}$$

from (37). The result of transformation is :

$$\nabla \cdot \left(i\partial_t + \nu_n \nabla^2 - \frac{1}{2}\omega_n \tilde{A} \tilde{N} \right) \tilde{E}_a = -\frac{1}{2}\omega_n \tilde{r}_a . \tag{86}$$

Note that, in view of

$$\langle \tilde{N} \tilde{\underline{E}}_a \rangle = 0$$

in a homogeneous medium, we may omit the operator $\tilde{A}$ in (86).

By integrating (86), we find

$$\left(i\partial_t + \nu_n \nabla^2 - \frac{1}{2}\omega_n \tilde{A}\tilde{N}\right)\tilde{\underset{\sim}{E}}_a = \tilde{\underset{\sim}{X}}_a \quad . \tag{87}$$

The forcing function is $\tilde{\underset{\sim}{X}}_a$, such that

$$\underset{\sim}{\nabla}\cdot\tilde{\underset{\sim}{X}}_a = -\frac{1}{2}\omega_n \tilde{n}_a \quad , \tag{88a}$$

or , upon integration,

$$\tilde{\underset{\sim}{X}}_a(t,\underset{\sim}{x}) = \frac{1}{2}\omega_n \underset{\sim}{\nabla}\frac{1}{4\pi}\int d\underset{\sim}{x}'\frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\tilde{n}_a(t,\underset{\sim}{x}') \tag{88b}$$

In the absence of the driving force, i.e. $\tilde{X}_a = 0$, (87) is degenerated into the first equation of the Zakharov equation

$$\left(i\partial_t + \nu_n \nabla^2 - \frac{1}{2}\omega_n \tilde{A}\tilde{N}\right)\tilde{\underset{\sim}{E}}_a = 0 \tag{89}$$

With the condition (2), the equation in the form (89) is also called the Schrödinger equation, as written in (1), and the equation in the form (87) will be called the forced Schrödinger equation.

6.2. <u>The second equation of the Zakharov system with the driving force</u>

The density $\tilde{N}$ is governed by the wave equation (44). By the use of the Poisson equation (83b), rewritten as

$$\underset{\sim}{\nabla}\, \underset{\sim}{\tilde{E}}_N = -\left(1 + \hat{e}_N/\hat{e}_n\right) \omega_N^2 \tilde{N} \; . \qquad (90)$$

by (25) and (81), we transform (44) into

$$\begin{aligned}\left(\partial_t^2 - c_s^2 \nabla^2\right)\tilde{N} &= 2\left(\tilde{Q}_M + \tilde{Q}_r\right) \\ &\quad + \left(1 + \hat{e}_n\right)\left(1 + \hat{e}_N/\hat{e}_n\right) \omega_N^2 \tilde{N} \\ &\cong 2\tilde{Q}_M = \nabla^2 \tilde{\xi} \\ &= \lambda c_s^2\, \nabla^2 \tilde{A}\, \left|\underset{\sim}{\tilde{E}}_a\right|^2 \end{aligned} \qquad (91)$$

by writing, from (60) :

$$\tilde{\xi} = \lambda c_s^2\, \tilde{A}\, \left|\underset{\sim}{\tilde{E}}_a\right|^2 \qquad (92)$$

with

$$\lambda = \left(d \xi c_s^2 \omega_n^2\right)^{-1} \qquad (93)$$

The approximation in (91) is made on the basis of the unequalities

$$\tilde{Q}_r \ll \tilde{r}$$

and

$$\omega_N \ll \omega_n$$

that are small by the factor $\xi^{-1} \ll 1$.

7. Conclusions

The hydrodynamics of solitons is governed by the system of equations (87) and (91) for the variables $\tilde{\underset{\sim}{E}}_a$ and $\tilde{N}$, respectively. The system contains the modulational nonlinearity by the ponderomotive force at the large scale and the nonlinear emission of sound by the strong velocity fluctuations at the small scale. The derivation is based upon finite fluctuations.

By the hypothesis of local nonlinearity that approximates (91) by (2), we reduce the system into the forced Schrödinger equation

$$\left(\iota\partial_t + \nu_n \nabla^2 + \frac{1}{2}\omega_n \lambda \tilde{A} \left|\tilde{\underset{\sim}{E}}_a\right|^2\right)\tilde{\underset{\sim}{E}}_a = \tilde{\underset{\sim}{X}}_a \tag{94}$$

The two nonlinearities can be clearly shown in the kinetic representation by the master equation

$$\left(\partial_t + \hat{L}\right)\hat{f}(t,\underset{\sim}{x},\underset{\sim}{E}) = 0 \,, \tag{95}$$

where the distribution function

$$\hat{f}(t,\underset{\sim}{x},\underset{\sim}{E}) = \delta\left[\underset{\sim}{E} - \hat{\underset{\sim}{E}}(t,\underset{\sim}{x})\right] \tag{96}$$

has $\underset{\sim}{E}$ as an independent random variable, and the differential operator

$$\hat{L} = -\iota\nu_n \nabla^2 + \frac{1}{2}\omega_n \hat{A}\tilde{N} - \iota\tilde{\underset{\sim}{X}}_a \underset{\sim}{\partial} \tag{97}$$

contains two nonlinearities from the modulation by $\tilde{N}$ and the sound emission by $\underset{\sim}{\tilde{X}}_a$, as two convections in the phase space.

If we confine ourselves to weak velocity fluctuations, we will miss the emission of sound by $\underset{\sim}{\tilde{X}}_a$, degenerating the system of equations (87) and (91) into the Zakharov equations and the equation (94) into the standard Schrödinger equation

$$\left(i\partial_t + \nu_r \nabla^2 + \tfrac{1}{2}\omega_n \lambda \tilde{A} \left| \underset{\sim}{\tilde{E}}_a \right|^2 \right) \underset{\sim}{\tilde{E}}_a = 0 \tag{98}$$

Our system of equations (87) and (91) and our forced Schrödinger equation (94) are valid for both plasma and fluid turbulence, since the derivation uses a general argument.

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15. SUPPLEMENTARY NOTES

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16. ABSTRACT

Theoretical and numerical works in atmospheric turbulence have used the Navier-Stokes fluid equations exclusively for describing large-scale motions. Controversy over the existence of an average temperature gradient for the very large eddies in the atmosphere suggested that a new theoretical basis for describing large-scale turbulence was necessary. A new soliton formalism as a fluid analogue that generalizes the Schrodinger equation and the Zakharov equations has been developed. This formalism, possessing all the nonlinearities including those from modulation provided by the density fluctuations and from convection due to the emission of finite sound waves by velocity fluctuations, treats large-scale turbulence as coalescing and colliding solitons. The new soliton system describes large-scale instabilities more explicitly than the Navier-Stokes system because it has a nonlinearity of the gradient type, while the Navier-Stokes has a nonlinearity of the non-gradient type. The forced Schrodinger equation for strong fluctuations describes the micro-hydrodynamical state of soliton turbulence and is valid for large-scale turbulence in fluids and plasmas where internal waves can interact with velocity fluctuations.

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